

The Center Variety of Polynomial Differential Systems

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Abstract

We investigate the symmetry component of the center variety of polynomial differential systems, corresponding to systems with an axis of symmetry in the real plane. We give a general algorithm to find this irreducible subvariety and compute its dimension. We show that our methods provide a simple way to compute the radical of the ideal generated by the focus quantities and, therefore, to estimate the cyclicity of a center in the case when the ideal is radical. In particular, we use our methods to get a simple proof of the famous Bautin theorem on the cyclicity of the quadratic system.

1. Introduction

We start with a brief history and background. For more information and proofs for the facts we state without proofs, see either the indicated original source or (2; 25; 26; 28; 29; 39).

In his Mémoire (19), Poincaré initiated the study of dynamical systems by studying real polynomial differential systems of the form

$$\begin{aligned}\frac{du}{dt} &= U(u, v), \\ \frac{dv}{dt} &= V(u, v),\end{aligned}\tag{1}$$

where U and V are polynomials over the real numbers with a singular point (u_0, v_0) , taken to be $(0, 0)$, without loss of generality. There he also defined many now standard concepts, in particular, the notion of a *center* of a system. The origin is a center if there exists a neighborhood \mathcal{U} of $(0, 0)$ such that every point of \mathcal{U} other than $(0, 0)$ is nonsingular, and the integral curve passing through that point is closed. Moreover, he proved the following theorem.

THEOREM 1: *Assume that the linearization of system (1) at the origin has purely imaginary eigenvalues. Then, without loss of generality, it is of the form*

$$\begin{aligned}\frac{du}{dt} &= -u + U_1(u, v) = U(u, v), \\ \frac{dv}{dt} &= v + V_1(u, v) = V(u, v),\end{aligned}\tag{2}$$

where U_1 and V_1 are polynomials with nonlinear terms. Then the origin is a center if and only if there exists a formal power series $\Psi(u, v) = u^2 + v^2 + \dots$, convergent in a neighborhood of the origin, such that

$$\frac{\partial \Psi}{\partial u} U + \frac{\partial \Psi}{\partial v} V \equiv 0.$$

The function $\Psi(u, v)$ is a local first integral of system (1).

Lyapunov (17) generalized and proved the above theorem for the case when P and Q are real analytic functions.

There are many different ways to enclose the set of plane real system (RS), into the set of two dimensional complex systems (CS). The most convenient and commonly used way is the following. Consider the real plane (u, v) as the complex line with the variable x :

$$x = u + iv.\tag{3}$$

Then System (1) is equivalent to the equation

$$i \frac{dx}{dt} = P(x, \bar{x}),\tag{4}$$

where $P = iU - V$. In many cases it is convenient to use just equation (4), however it is natural to add to this equation its complex conjugate, $-i\dot{\bar{x}} = \bar{P}(x, \bar{x})$ and consider $y = \bar{x}$ as a new variable and $Q = \bar{P}$ as a new function. As a result, we get the system of two complex differential equations

$$i\dot{x} = P(x, y), \quad -i\dot{y} = Q(x, y).\tag{5}$$

Thus there is one-to-one correspondence between systems from (RS) and the subset of (CS) consisting of systems of the form (5) where the second equation is the complex conjugate to the first one.

For polynomial systems of the form (2) the procedure above yields the system

$$\begin{aligned}i \frac{dx}{dt} &= x + P_1(x, y) = P(x, y), \\ -i \frac{dy}{dt} &= y + Q_1(x, y) = Q(x, y),\end{aligned}\tag{6}$$

where P_1 and Q_1 are complex polynomials with nonlinear terms and $Q_1 = \overline{P_1}$. After the change of time $id\tau = dt$ we can write (6) in the form

$$\begin{aligned}\frac{dx}{d\tau} &= x + P_1(x, y) = P(x, y), \\ -\frac{dy}{d\tau} &= y + Q_1(x, y) = Q(x, y).\end{aligned}\tag{7}$$

In (10), Dulac considered system (7) where P_1 and Q_1 are arbitrary complex polynomials with nonlinear terms. Moreover, he gave the following definition for a center at the origin.

DEFINITION 1: *System (7) has a center at the origin if there is an analytic first integral of the form*

$$\Psi(x, y) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^s v_{j,s-j} x^j y^{s-j},\tag{8}$$

where the $v_{j,s-j}$ are functions in the coefficients of P and Q .

If (7) is the complexification of (2) by means of (3) then this definition is in agreement with the definition given by Poincaré.

When P and Q are quadratic polynomials, Dulac gave necessary and sufficient conditions on the coefficients of P and Q such that system (7) has a center at the origin. Moreover, he asked if one can find necessary and sufficient conditions on the coefficients of P and Q (of any given degree) such that system (7) has a center at the origin. This is the so-called *center problem*.

In this paper, we present a partial solution to the center problem for polynomial systems, using methods from computational algebra.

Any polynomial system of the form (7) can be written in the form

$$\begin{aligned}\frac{dx}{d\tau} &= x - \sum_{(p,q) \in S} a_{pq} x^{p+1} y^q = P(x, y), \\ -\frac{dy}{d\tau} &= y - \sum_{(p,q) \in S} b_{qp} x^q y^{p+1} = -Q(x, y),\end{aligned}\tag{9}$$

where $P(x, y), Q(x, y) \in \mathbb{C}[x, y]$ and

$$S = \{(p_i, q_i) \mid p_i + q_i \geq 1, i = 1, \dots, l\} \subset \{\{-1\} \cup \mathbb{N}\} \times \mathbb{N}.$$

Throughout this paper, \mathbb{N} is the set of nonnegative integers.

We denote by $E(a, b) (= \mathbb{C}^{2l})$ the parameter space of (9), and by $\mathbb{C}[a, b]$ the polynomial ring in the variables a_{pq}, b_{qp} .

As we have shown above, in the case when

$$y = \bar{x}, b_{ij} = \bar{a}_{ji}, id\tau = dt\tag{10}$$

system (9) is equivalent to the system

$$i \frac{dx}{dt} = x - \sum_{(p,q) \in S} a_{pq} x^{p+1} \bar{x}^q, \quad (11)$$

which has a center or focus at the origin in the real plane $\{(u, v) \mid x = u + iv\}$, where the system can be also written in the form (2)

$$\dot{u} = -v + U_1(u, v), \quad \dot{v} = u + V_1(u, v).$$

In this case we denote the parameter space by $E(a)$.

DEFINITION 2: *We say that System (9) has a center on the set $W \subset E(a, b)$, if for any point $(\tilde{a}, \tilde{b}) \in W$ the corresponding system (9) has a center at the origin.*

It is known (10) that one can always find a Lyapunov function Ψ of the form (8) such that

$$\frac{\partial \Psi}{\partial x} P(x, y) + \frac{\partial \Psi}{\partial y} Q(x, y) = g_{11} \cdot (xy)^2 + g_{22} \cdot (xy)^3 + g_{33} \cdot (xy)^4 + \dots, \quad (12)$$

where the g_{ii} are polynomials of $\mathbb{C}[a, b]$ called *focus quantities*. Thus, the maximal set $V \subset E(a, b)$, on which System (9) has a center, is the set where all polynomials $g_{ii}, i = 1, 2, \dots$, vanish, that is, V is the variety of the ideal generated by the focus quantities g_{ii} .

Denote by $\mathbf{V}(I)$ the variety of the ideal I .

DEFINITION 3: *The set*

$$V = \mathbf{V}(\langle g_{11}, g_{22}, \dots, g_{ii}, \dots \rangle)$$

is called the center variety of system (9).

So, for every point in V the corresponding system has a center at the origin in the sense that there is a first integral of the form (8). However, if $(a, b) \in V$ and $a_{pq} = \bar{b}_{qp}$ for all $(p, q) \in S$, then such a point corresponds to a real system of the form (11), which then has a topological center at the origin in the plane $x = u + iv$. (For a geometrical interpretation of the center of the complex system (9) see, e.g., (39).)

Therefore, given a system of the form (9), the problem of finding the center variety (center problem) of the system arises.

Among the components of the center variety of polynomial system (9) there are at least two components which can be found without computing any focus quantities: one component consists of Hamiltonian systems, and the other one, which we call the *symmetry component*, corresponds to systems which have an axis of symmetry in the real plane $x = u + iv$. The component of Hamiltonian

systems has a simple geometry because it is equal to the intersection of linear subspaces of $E(a, b)$.

To find the symmetry components one can proceed as follows. With system (9) we associate the linear operator

$$\begin{aligned} L(\nu) &= \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix} \\ &= \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \nu_1 + \cdots + \begin{pmatrix} p_l \\ q_l \end{pmatrix} \nu_l + \begin{pmatrix} q_l \\ p_l \end{pmatrix} \nu_{l+1} + \cdots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \nu_{2l}, \end{aligned} \quad (13)$$

where $(p_m, q_m) \in S$. Let \mathcal{M} denote the set of all solutions $\nu = (\nu_1, \nu_2, \dots, \nu_{2l})$ with non-negative components of the equation

$$L(\nu) = \begin{pmatrix} k \\ k \end{pmatrix}, \quad (14)$$

for all $k \in \mathbb{N}$. Obviously, \mathcal{M} is an Abelian monoid. For $k = 0$, finding the solutions of equation (14) is just the standard integer programming problem, see, e.g., (34, Sect. 1.4). Our algorithm is an adaptation of the algorithm to solve the standard problem. Let $\mathbb{C}[\mathcal{M}]$ denote the subalgebra of $\mathbb{C}[a, b]$ generated by all monomials of the form

$$a_{p_1 q_1}^{\nu_1} a_{p_2 q_2}^{\nu_2} \cdots a_{p_l q_l}^{\nu_l} b_{q_l p_l}^{\nu_{l+1}} b_{q_{l-1} p_{l-1}}^{\nu_{l+2}} \cdots b_{q_1 p_1}^{\nu_{2l}},$$

for all $\nu \in \mathcal{M}$. In order to simplify notation we will abbreviate such a monomial by $[\nu] = [\nu_1, \dots, \nu_{2l}]$. For $\nu \in \mathcal{M}$, let

$$\bar{\nu} = (\nu_{2l}, \nu_{2l-1}, \dots, \nu_1).$$

Furthermore, let

$$\text{IM}[\nu] = [\nu] - [\bar{\nu}], \quad \text{RE}[\nu] = [\nu] + [\bar{\nu}].$$

It is shown in (21; 23) that the focus quantities of system (9) belong to $\mathbb{C}[\mathcal{M}]$ and have the form

$$g_{kk} = \sum_{L(\nu)=(k,k)^t} g_{(\nu)} \text{IM}[\nu], \quad (15)$$

with $g_{(\nu)} \in \mathbf{Q}$, $k = 1, 2, \dots$ (Similar properties of the focus quantities were also obtained in (4; 36).)

Consider the ideal

$$I_{sym} = \langle [\nu] - [\bar{\nu}] \mid \nu \in \mathcal{M} \rangle \subset \mathbb{C}[\mathcal{M}],$$

called the *Sibirsky ideal* of (9). The next statement is an obvious generalization of Sibirsky's symmetry criteria (31) for a center for real systems to systems of the form (9).

PROPOSITION 1: *The system (9) has a center on the set $\mathbf{V}(I_{sym})$.*

DEFINITION 4: *The set $\mathbf{V}(I_{sym})$ is called the symmetry component of the center variety.*

For points $(a, b) \in \mathbf{V}(I_{sym})$ for which $a_{pq} = \bar{b}_{qp}$ the corresponding systems (11) have an axis of symmetry (see Section 3). This is the reason for the name “symmetry component.” However we should mention that we do not know a proof that $\mathbf{V}(I_{sym})$ is indeed a component of the center variety, that is, it is not a proper subvariety of an irreducible subvariety of the center variety. However Proposition 1 implies that $\mathbf{V}(I_{sym})$ is a subset of the center variety and we shall show below, that $\mathbf{V}(I_{sym})$ is irreducible. Moreover, for all cubic polynomial systems $\mathbf{V}(I_{sym})$ we have investigated, it is indeed a component. Thus we conjecture that for any polynomial system of the form (9), $\mathbf{V}(I_{sym})$ is a component of the center variety.

In the present paper we give a simple and effective algorithm for computing generators for the Sibirsky ideal, using methods from computational algebra. This allows us to find a finite set of defining polynomials for the symmetry component of the center variety. Moreover, we prove the following results.

THEOREM 2: *The Sibirsky ideal I_{sym} is prime in $\mathbb{C}[a, b]$.*

THEOREM 3: *The dimension of the symmetry component, $\mathbf{V}(I_{sym})$, of the center variety is equal to l if all coefficients on the right-hand side of system (9) are resonant and $l + 1$ otherwise.*

Recall that the resonant coefficients are the ones which cannot be canceled by transformation of the system to the Poincaré-Lyapunov normal form, that is, the coefficients of the form a_{kk}, b_{kk} .

It should be mentioned that a substantial amount of literature is devoted to different particular subfamilies of polynomial systems, mainly to systems of the second to fifth degree (the bibliography on quadratic systems alone by J. Reyn (20) contains approximately 1500 references). However, the Symmetry Component Algorithm in Section 2, and Theorems 2 and 3 are among the very few statements known up to now about the whole class of polynomial systems.

The center problem is closely connected to the *cyclicity* problem, which is sometimes called the local 16th Hilbert problem (14).

DEFINITION 5: *Let $n_{a,\epsilon}$ be the number of limit cycles of system (11) in an ϵ -neighborhood of the origin. Then we say that the singular point $x = 0$ of system (11) with given coefficients $a^* \in E(a)$ has cyclicity k with respect to the space $E(a)$, if there exist δ_0, ϵ_0 such that for every $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta_0$*

$$\max_{a \in U_\delta(a^*)} n_{a,\epsilon} = k.$$

The cyclicity of the origin of real quadratic systems (that is, system (11) with quadratic non-linearities) was first investigated by Bautin (3), and by Sibirsky (30) (see also (38)) for the system (11) with homogeneous cubic nonlinearities. They proved that the cyclicity of the systems is less or equal two, respectively four. (If we take into account perturbations by linear terms, then the cyclicity is 3 and 5, respectively).

In this paper we show that Theorem 2 along with other methods from computational algebra provides an efficient tool to investigate the cyclicity of polynomial systems in those cases, where the ideal of focus quantities $\langle g_{11}, g_{22}, \dots \rangle$ is radical. Moreover, even when the ideal of focus quantities is not radical, we use our algorithm along with some standard methods to compute the cyclicity of some cubic systems, see (16).

2. An Algorithm for the Symmetry Component

In this section we give an algorithm to find a finite set of generators for the Sibirsky ideal, hence for the symmetry component of the center variety. It works for general systems (9). As a corollary, we obtain a Hilbert basis, that is, a finite minimal generating set, for the monoid \mathcal{M} described in the introduction.

Let

$$\mathcal{A} = \begin{bmatrix} c_{11} & c_{21} & \dots & c_{l1} & c_{l2} & \dots & c_{22} & c_{12} \\ c_{12} & c_{22} & \dots & c_{l2} & c_{l1} & \dots & c_{21} & c_{11} \end{bmatrix}$$

be a $(2 \times 2l)$ -matrix with entries $c_{ij} \in \mathbb{Z}$. For each $k \in \mathbb{N}$, let

$$\mathcal{V}(k) = \left\{ \nu = (\nu_1, \dots, \nu_{2l}) \in \mathbb{N}^{2l} \mid \mathcal{A} \cdot \nu^t = \begin{pmatrix} k \\ k \end{pmatrix} \right\}.$$

The proof of the following lemma is straightforward.

LEMMA 1: *Let $\mathcal{M}_{\mathcal{A}} = \cup_{k \in \mathbb{N}} \mathcal{V}(k)$. Then $\mathcal{M}_{\mathcal{A}}$ is an Abelian submonoid of \mathbb{N}^{2l} . Also, if*

$$\mu = (\mu_1, \dots, \mu_{2l}) \in \mathcal{M}_{\mathcal{A}},$$

then $\bar{\mu} = (\mu_{2l}, \dots, \mu_1) \in \mathcal{M}_{\mathcal{A}}$.

Let $d = 2l$ and $R = \mathbb{C}[x_1, \dots, x_d]$ be the polynomial ring in d variables. We consider the following binomial ideal in R :

$$I_{\mathcal{A}} = \langle \mathbf{x}^{\mu} - \mathbf{x}^{\bar{\mu}} \mid \mu \in \mathcal{M}_{\mathcal{A}} \rangle.$$

We will obtain a Hilbert basis for $\mathcal{M}_{\mathcal{A}}$ from a Gröbner basis of the ideal $I_{\mathcal{A}}$. If the matrix \mathcal{A} arises from the coefficients of system (1) as follows:

$$\mathcal{A} = \begin{pmatrix} p_1 & p_2 & \dots & p_l & q_l & q_{l-1} & \dots & q_1 \\ q_1 & q_2 & \dots & q_l & p_l & p_{l-1} & \dots & p_1 \end{pmatrix},$$

then $I_{\mathcal{A}}$ is precisely the Sibirsky ideal of (9).

We will represent $I_{\mathcal{A}}$ as the kernel of a homomorphism of polynomial rings, so we can use a standard algorithm to compute a finite generating set for this ideal. As a consequence, we obtain an algorithm to compute a finite set of polynomials that define the symmetry component of the center variety of (1).

Let $S = \mathbb{C}[t_1^{\pm}, t_2^{\pm}, y_1, \dots, y_l]$. Define a ring homomorphism $\phi : R \longrightarrow S$ by

$$\begin{aligned} x_i &\longmapsto y_i t_1^{c_{i1}} t_2^{c_{i2}} \\ x_{l+i} &\longmapsto y_{l-i+1} t_1^{c_{l-i+1,2}} t_2^{c_{l-i+1,1}} \end{aligned}$$

for $i = 1, \dots, l$.

THEOREM 4: $\ker(\phi) = I_{\mathcal{A}}$. In particular, $I_{\mathcal{A}}$ is a prime ideal in R .

Proof: The second statement follows immediately from the first, since S is a domain.

To prove that $\ker(\phi) = I_{\mathcal{A}}$, we first show that $\ker(\phi)$ is a binomial ideal. We can factor the map ϕ as follows. Let

$$T = k[t_1, t_2, s_1, s_2, x_1, \dots, x_d, y_1, \dots, y_l],$$

and

$$\phi' : \mathbb{C}[x_1, \dots, x_d] \longrightarrow T$$

be defined like ϕ , except that if $c_{ij} < 0$, then it appears as an exponent of the variable s_j instead of t_j . Then ϕ is equal to the composition of ϕ' followed by the projection

$$T \longrightarrow T / \langle t_1 s_1 - 1, t_2 s_2 - 1 \rangle.$$

It is straightforward to see that $\ker(\phi) = \ker(\phi')$. Let

$$J = \langle x_i - y_i t_1^{c_{i1}} t_2^{c_{i2}}, x_{l+i} - y_{l-i+1} t_1^{c_{l-i+1,2}} t_2^{c_{l-i+1,1}} \mid i = 1, \dots, l \rangle \subset T.$$

Then it follows immediately from (1, Theorem 2.4.2) that $\ker(\phi) = \ker(\phi') = J \cap R$. We obtain a generating set for $J \cap R$ by computing a reduced Gröbner basis for J using an elimination ordering with $x_i < y_j, t_k, s_r$ for all i, j, k, r , and then intersecting it with R . Since J is generated by binomials, any reduced Gröbner basis also consists of binomials. This shows that $\ker(\phi)$ is a binomial ideal.

Now, let $\mathbf{x}^\alpha - \mathbf{x}^\beta$ be a binomial in R . We may assume that the two monomials have no common factors, that is, $\text{supp}(\alpha) \cap \text{supp}(\beta) = \emptyset$. Then

$$\begin{aligned} \phi(\mathbf{x}^\alpha - \mathbf{x}^\beta) &= (y_1 t_1^{c_{11}} t_2^{c_{12}})^{\alpha_1} \dots (y_l t_1^{c_{l1}} t_2^{c_{l2}})^{\alpha_l} (y_l t_1^{c_{l2}} t_2^{c_{l1}})^{\alpha_{l+1}} \dots (y_1 t_1^{c_{12}} t_2^{c_{11}})^{\alpha_d} \\ &\quad - (y_1 t_1^{c_{11}} t_2^{c_{12}})^{\beta_1} \dots (y_l t_1^{c_{l1}} t_2^{c_{l2}})^{\beta_l} (y_l t_1^{c_{l2}} t_2^{c_{l1}})^{\beta_{l+1}} \dots (y_1 t_1^{c_{12}} t_2^{c_{11}})^{\beta_d} \\ &= y_1^{(\alpha_1 + \alpha_{2l})} \dots y_l^{(\alpha_l + \alpha_{l+1})} t_1^{(c_{11}\alpha_1 + \dots + c_{l1}\alpha_l + c_{l2}\alpha_{l+1} + \dots + c_{12}\alpha_d)} t_2^{(c_{12}\alpha_1 + \dots + c_{l2}\alpha_l + c_{l1}\alpha_{l+1} + \dots + c_{11}\alpha_d)} \\ &\quad - y_1^{(\beta_1 + \beta_{2l})} \dots y_l^{(\beta_l + \beta_{l+1})} t_1^{(c_{11}\beta_1 + \dots + c_{l1}\beta_l + c_{l2}\beta_{l+1} + \dots + c_{12}\beta_d)} t_2^{(c_{12}\beta_1 + \dots + c_{l2}\beta_l + c_{l1}\beta_{l+1} + \dots + c_{11}\beta_d)} \end{aligned}$$

Thus, $\phi(\mathbf{x}^\alpha - \mathbf{x}^\beta) = 0$ if and only if

$$\text{for all } i \geq 1, \alpha_i + \alpha_{d-i+1} = \beta_i + \beta_{d-i+1},$$

$$\begin{aligned} c_{11}\alpha_1 + \cdots + c_{l1}\alpha_l + c_{l2}\alpha_{l+1} &+ \cdots + c_{12}\alpha_d \\ &= c_{11}\beta_1 + \cdots + c_{l1}\beta_l + c_{l2}\beta_{l+1} + \cdots + c_{12}\beta_d, \end{aligned}$$

and

$$\begin{aligned} c_{12}\alpha_1 + \cdots + c_{l2}\alpha_l + c_{l1}\alpha_{l+1} &+ \cdots + c_{11}\alpha_d \\ &= c_{12}\beta_1 + \cdots + c_{l2}\beta_l + c_{l1}\beta_{l+1} + \cdots + c_{11}\beta_d. \end{aligned}$$

Since $\text{supp}(\alpha) \cap \text{supp}(\beta) = \emptyset$, we obtain the following facts from the first condition in (16). If, for some i , $\alpha_i = 0 = \beta_i$, then $\alpha_{d-i+1} = 0 = \beta_{d-i+1}$, so that $\beta_i = \alpha_{d-i+1}$. Now suppose that $\beta_i \neq 0$ for some i . Then $\alpha_i = 0$. But then $\beta_{d-i+1} = 0$, otherwise $\alpha_{d-i+1} = 0$, which cannot be since $\beta_i \geq 0$ for all i . And this, in turn, implies that $\beta_i = \alpha_{d-i+1}$. Hence

$$\beta = (\beta_1, \dots, \beta_d) = (\alpha_d, \dots, \alpha_1) = \overline{\alpha}.$$

It follows from the last two equations in (16) that $\alpha \in \mathcal{M}_A$. This completes the proof of the theorem. \square

THEOREM 5: *With notation as above, let G be a reduced Gröbner basis of I_A , with respect to some term ordering on R . Then*

1. *The set*

$$\mathcal{H} = \{\mu, \overline{\mu} \mid \mathbf{x}^\mu - \mathbf{x}^{\overline{\mu}} \in G\} \cup \{\mathbf{e}_i + \mathbf{e}_j \mid j = d - i + 1; i = 1, \dots, l\},$$

where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the i th basis vector, is a Hilbert basis of \mathcal{M}_A . Note that $\mathbf{e}_i + \mathbf{e}_{d-i+1} \in \mathcal{M}_A$ for all i .

2. $I_A = \langle \mathbf{x}^\mu - \mathbf{x}^{\overline{\mu}} \mid \mu \in \mathcal{H} \rangle$.

Proof: The proof is similar to that of (34, Algorithm 1.4.5). It was shown in the proof of the previous theorem that any binomial in I_A is of the form $\mathbf{x}^\mu - \mathbf{x}^{\overline{\mu}}$ for some $\mu \in \mathcal{M}_A$. Hence the Gröbner basis G of I_A is of the form

$$G = \{\mathbf{x}^{\mu_1} - \mathbf{x}^{\overline{\mu_1}}, \dots, \mathbf{x}^{\mu_r} - \mathbf{x}^{\overline{\mu_r}}\}.$$

We first show that the set \mathcal{H} is a generating set for \mathcal{M}_A . Suppose not, so there exists $\mu \in \mathcal{M}_A$ which is not an \mathbb{N} -linear combination of elements in \mathcal{H} . We can choose μ so that \mathbf{x}^μ is minimal with respect to the chosen term ordering of R . Since $\mu \in \mathcal{H}$, we have $\phi(\mathbf{x}^\mu - \mathbf{x}^{\overline{\mu}}) = 0$, so that $\mathbf{x}^\mu - \mathbf{x}^{\overline{\mu}} \in I_A$. Hence the leading

term of this binomial is divisible by a binomial in G . That is, there exist i, β such that

$$\mathbf{x}^\mu - \mathbf{x}^{\bar{\mu}} = \mathbf{x}^\beta (\mathbf{x}^{\mu_i} - \mathbf{x}^{\bar{\mu}_i}) + \mathbf{x}^{\bar{\mu}_i} (\mathbf{x}^\beta - \mathbf{x}^{\bar{\beta}}).$$

Thus $\mathbf{x}^{\bar{\mu}_i} (\mathbf{x}^\beta - \mathbf{x}^{\bar{\beta}}) \in I_{\mathcal{A}}$. But $I_{\mathcal{A}}$ is a prime ideal, and it is immediate from the definition of ϕ that it contains no monomials, so $(\mathbf{x}^\beta - \mathbf{x}^{\bar{\beta}}) \in I_{\mathcal{A}}$. Moreover, $\mathbf{x}^\beta < \mathbf{x}^\mu$. Hence, by the choice of μ , β is a linear combination of elements of \mathcal{H} , which implies that $\mu = \beta + \mu_i$ is also a linear combination of elements of \mathcal{H} . This is a contradiction to our assumption on μ . Thus, \mathcal{H} is a generating set for $\mathcal{M}_{\mathcal{A}}$.

To show that \mathcal{H} is minimal, suppose that some μ_i or $\bar{\mu}_i$ is an \mathbb{N} -linear combination of elements in \mathcal{H} . Since the Gröbner basis G is reduced, the linear combination cannot contain any summands coming from the $\mu_i, \bar{\mu}_i$. But observe that all the vectors $\mathbf{e}_i + \mathbf{e}_j$ are symmetric, so that

$$\mathbf{x}^{\mathbf{e}_i + \mathbf{e}_j} - \mathbf{x}^{\overline{\mathbf{e}_i + \mathbf{e}_j}} = 0.$$

A similar argument disposes of the other cases. Thus, we have shown that \mathcal{H} is the Hilbert basis of $\mathcal{M}_{\mathcal{A}}$.

$$2. \langle \mathbf{x}^\mu - \mathbf{x}^{\bar{\mu}} \mid \mu \in \mathcal{H} \rangle = \langle G \rangle = I_{\mathcal{A}}. \quad \square$$

We can also compute the dimension of the affine variety $\mathbf{V}(I_{\mathcal{A}})$ of the ideal $I_{\mathcal{A}}$.

THEOREM 6: *The dimension of $\mathbf{V}(I_{\mathcal{A}})$ is equal to l if $c_{i1} = c_{i2}$ for all $i = 1, \dots, l$, and $l + 1$ otherwise.*

Proof: According to (33, Lemma 4.2) the dimension of $\mathbf{V}(I_{\mathcal{A}})$ is equal to the number of linearly independent column vectors in the matrix

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 & \dots & 0 & 0 \\ c_{11} & c_{21} & \dots & c_{l1} & c_{l2} & \dots & c_{22} & c_{12} \\ c_{12} & c_{22} & \dots & c_{l2} & c_{l1} & \dots & c_{21} & c_{11} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & c_{l2} - c_{l1} & \dots & c_{22} - c_{21} & c_{12} - c_{11} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

The theorem now follows. \square

Proof of Theorems 2 and 3. Theorem 2 is a corollary of Theorems 4 and 5 and Theorem 3 follows immediately from Theorem 6.

To close this section, we summarize the algorithm, and in the next section we shall provide several examples.

Symmetry Component Algorithm

Input: Two sequences of integers $p_1, \dots, p_l, p_i \geq -1; q_1, \dots, q_l, q_i \geq 0$. (These are the coefficient labels for a system of the form (9).)

Output: A finite set of generators for the Sibirsky ideal I_{sym} of (9), and the Hilbert basis \mathcal{H} of the monoid \mathcal{M} .

1. Compute a reduced Gröbner basis G for the ideal

$$\mathcal{J} = \langle a_{p_i q_i} - y_i t_1^{p_i} t_2^{q_i}, b_{q_i p_i} - y_{l-i+1} t_1^{q_{l-i+1}} t_2^{p_{l-i+1}} \mid i = 1, \dots, l \rangle$$

in $\mathbb{C}[a, b, y_1, \dots, y_l, t_1^\pm, t_2^\pm]$, with respect to any elimination ordering with

$$\{t_1, t_2\} > \{y_1, \dots, y_d\} > \{a_{p_1 q_1}, \dots, b_{q_1 p_1}\}.$$

2. $I_{sym} = \langle G \cap \mathbb{C}[a, b] \rangle$.
3. $\mathcal{H} = \{\nu, \bar{\nu} \mid [\nu] - [\bar{\nu}] \in G\} \cup \{\mathbf{e}_i + \mathbf{e}_j \mid j = d - i + 1; i = 1, \dots, l\}$ is a Hilbert basis for the monoid \mathcal{M} , where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ is the i th basis vector.

3. Invariants of the Rotation Group and the Symmetry Components

The symmetry component of the center variety of a real system in the complex form (11) was investigated by Sibirsky and his coworkers (31; 32), using invariants of the rotation group of this system.

He showed that a monomial

$$a_{p_1 q_1}^{\nu_1} a_{p_2 q_2}^{\nu_2} \cdots a_{p_l q_l}^{\nu_l} \bar{a}_{q_l p_l}^{\nu_{l+1}} \cdots \bar{a}_{q_2 p_2}^{\nu_{2l-1}} \bar{a}_{q_1 p_1}^{\nu_{2l}}$$

is invariant under the action of the rotation group

$$x \longmapsto x e^{i\varphi} \tag{16}$$

if and only if $\nu \in \mathbb{N}^{2l}$ is a solution of the Diophantine equation

$$\begin{aligned} (p_1 - q_1)\nu_1 + (p_2 - q_2)\nu_2 &+ \cdots + (p_l - q_l)\nu_l \\ &+ (q_l - p_l)\nu_{l+1} + \cdots + (q_1 - p_1)\nu_{2l} = 0, \end{aligned} \tag{17}$$

which is obtained by subtracting the second equation of system (14) from the first one.

PROPOSITION 2: *The set of all non-negative integer solutions of equation (17) coincides with the monoid \mathcal{M} defined by equation (14).*

Proof: Obviously, every solution of (14) is also a solution of (17). Conversely, let ν be a solution of (17). Then

$$L^1(\nu) = L^2(\nu) = k \quad (18)$$

and

$$L^1(\mathbf{e}_i) + L^2(\mathbf{e}_i) = L^1(\mathbf{e}_{2l-i}) + L^2(\mathbf{e}_{2l-i}) = p_i + q_i \geq 1, \quad (19)$$

for $i = 1, \dots, l$. Taking into account that $L(\nu)$ is a linear operator, we conclude that k on the right-hand side of (18) is non-negative. \square

Thus, to find invariants of system (11) under the rotation group (16) it is sufficient to find a generating set of the monoid \mathcal{M} . It also is easily seen that the monomials $[\nu], \nu \in \mathcal{M}$ are invariants of the system (9) under the action of the transformation

$$x \longmapsto xe^{i\varphi}, \quad y \longmapsto ye^{-i\varphi}.$$

Knowledge of the invariants of system (11) allows one to determine whether the corresponding vector field has an axis of symmetry. Denote by \tilde{E} the subset of $E(a)$ consisting of systems (11) such that the fraction $U(u, v)/V(u, v)$ is irreducible, where U, V are the functions on the right-hand side of system (2). The following theorems are proven in (31).

THEOREM 7: *If $a^* \in E(a)$ and $\text{IM}[\nu] = 0$ at the point (a^*, \bar{a}^*) for all $\nu \in \mathcal{M}$, then the corresponding vector field (2) has an axis of symmetry passing through the origin. Moreover, if $a^* \in \tilde{E}$ then the opposite statement holds.*

According to this theorem, for every point in $\mathbf{V}(I_{\text{sym}})$ of the form (a^*, \bar{a}^*) the corresponding vector field (2) has an axis of symmetry passing through the origin. This is the reason for calling $\mathbf{V}(I_{\text{sym}})$ the “symmetry component.” However, we have not investigated whether for arbitrary points in $\mathbf{V}(I_{\text{sym}})$ the corresponding systems (9) have any kind of symmetry.

As an immediate corollary of Theorem 7 we get

THEOREM 8: *If $a^* \in E(a)$ and $\text{IM}[\nu] = 0$ at the point (a^*, \bar{a}^*) for all $\nu \in M$, then the corresponding vector field (2) has a center at the origin.*

Similarly, it follows from Proposition 1, that if

$$\text{IM}[\nu] = 0 \quad (20)$$

for all $\nu \in M$ at $(a^*, b^*) \in E(a, b)$, then the corresponding system (9) has a center at the origin.

As mentioned above, according to (31; 32) the monomial $[\nu]$ is an invariant of the rotation group (16) if and only if ν is a solution of (17). Proposition 2 implies that the set of solutions of equation (17) coincides with \mathcal{M} . The invariant $[\nu]$ is called *irreducible* if ν cannot be written in the form $\nu = \mu + \theta$, where μ, θ are

invariants. Therefore the set $\{[\nu_i]\}$ is the set of all irreducible invariants if and only if $\{\nu_i\}$ is the Hilbert basis of \mathcal{M} .

It is proven in (32), that the maximal degree of irreducible invariants of the group (16) is less than or equal to

$$2(1 + \max_{(p,q) \in S} (p + q)). \quad (21)$$

Using this and equation (14) or (17) it is possible to find a generating set of the monoid \mathcal{M} by sorting. In particular, in (31) center conditions are obtained for the cubic system

$$i \frac{dx}{dt} = x(1 - a_{10}x - a_{01}\bar{x} - a_{-12}x^{-1}\bar{x}^2 - a_{20}x^2 - a_{11}x\bar{x} - a_{02}\bar{x}^2 - a_{-13}x^{-1}\bar{x}^3) \quad (22)$$

which can also be obtained from Theorem 9 below by taking imaginary parts.

Theorem 5 provides another way to find a Hilbert basis of the monoid \mathcal{M} and, therefore, the center conditions (20). We apply it to find the symmetry component of the general cubic system

$$\begin{aligned} \dot{x} &= x(1 - a_{10}x - a_{01}y - a_{-12}x^{-1}y^2 - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3), \\ \dot{y} &= -y(1 - b_{2,-1}x^2y^{-1} - b_{10}x - b_{01}y - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{11}xy - b_{02}y^2), \end{aligned} \quad (23)$$

where $x, y, a_{ij}, b_{ij} \in \mathbb{C}$.

THEOREM 9: *The symmetry component V_{sym} of the center variety of cubic system (23) is defined by the following equations:*

$$\begin{aligned} 0 &= a_{11} - b_{11} = a_{01}b_{02}b_{2,-1} - a_{-1,2}b_{10}a_{20} = a_{01}a_{02}b_{2,-1} - a_{-1,2}b_{10}b_{20} \\ &= a_{10}^4a_{-13} - b_{3,-1}b_{01}^4 = a_{10}a_{-12}b_{20} - b_{01}b_{2,-1}a_{02} = a_{10}a_{-12}b_{10}^2 - a_{01}^2b_{2,-1}b_{01} \\ &= a_{20}a_{02} - b_{20}b_{02} = a_{10}^2a_{-1,2}b_{10} - a_{01}b_{2,-1}b_{01}^2 = a_{10}b_{02}b_{10} - a_{01}a_{20}b_{01} \\ &= a_{01}^3b_{2,-1} - a_{-12}b_{10}^3 = a_{10}a_{02}b_{10} - a_{01}b_{20}b_{01} = a_{10}^3a_{-12} - b_{2,-1}b_{01}^3 \\ &= a_{10}a_{-1,3}b_{2,-1} - a_{-12}b_{3,-1}b_{01} = a_{20}a_{-1,3}b_{20} - a_{02}b_{3,-1}b_{02} = a_{10}^2b_{02} - a_{20}b_{01}^2 \\ &= a_{02}^2b_{3,-1} - a_{-13}b_{20}^2 = a_{01}a_{-12}b_{3,-1} - a_{-13}b_{2,-1}b_{10} = a_{01}^2b_{20} - a_{02}b_{10}^2 \\ &= a_{20}^2a_{-13} - b_{3,-1}b_{02}^2 = a_{10}a_{-13}b_{20}b_{10} - a_{01}a_{02}b_{3,-1}b_{01} \\ &= a_{10}a_{20}a_{-1,3}b_{10} - a_{01}b_{3,-1}b_{02}b_{01} = a_{10}b_{02}^2b_{2,-1} - a_{-12}a_{20}^2b_{01} \\ &= a_{10}^2a_{02} - b_{20}b_{01}^2 = a_{10}a_{02}b_{02}b_{2,-1} - a_{-12}a_{20}b_{20}b_{01} \\ &= a_{10}a_{02}^2b_{2,-1} - a_{-12}b_{20}^2b_{01} = a_{01}^2b_{3,-1}b_{02} - a_{20}a_{-13}b_{10}^2 = a_{01}^2a_{20} - b_{02}b_{10}^2 \\ &= a_{01}a_{-12}b_{20}^2 - a_{02}^2b_{2,-1}b_{10} = a_{10}^2a_{-13}b_{20} - a_{02}b_{3,-1}b_{01}^2 = a_{-12}a_{20}b_{10} - b_{02}b_{2,-1}b_{01} \\ &= a_{01}a_{-12}a_{20}b_{20} - a_{02}b_{02}b_{2,-1}b_{10} = a_{01}^2a_{02}b_{3,-1} - a_{-13}b_{20}b_{10}^2 \\ &= a_{10}^2a_{20}a_{-13} - b_{3,-1}b_{02}b_{01}^2 = a_{01}a_{-12}a_{20}^2 - b_{02}^2b_{2,-1}b_{10} = a_{10}a_{-13}b_{10}^3 - a_{01}^3b_{3,-1}b_{01} \\ &= a_{10}^2a_{-13}b_{10}^2 - a_{01}^2b_{3,-1}b_{01}^2 = a_{10}^3a_{-13}b_{10} - a_{01}b_{3,-1}b_{01}^3 = a_{01}^4b_{3,-1} - a_{-13}b_{10}^4 \\ &= a_{01}a_{-13}b_{20}b_{2,-1} - a_{-12}a_{02}b_{3,-1}b_{10} = a_{01}a_{20}a_{-13}b_{2,-1} - a_{-12}b_{3,-1}b_{02}b_{10} \\ &= a_{10}a_{-12}b_{3,-1}b_{02} - a_{20}a_{-13}b_{2,-1}b_{01} = a_{10}a_{-12}a_{02}b_{3,-1} - a_{-13}b_{20}b_{2,-1}b_{01} \end{aligned} \quad (24)$$

$$\begin{aligned}
&= a_{-12}^2 b_{3,-1} b_{20} - a_{02} a_{-13} b_{2,-1}^2 = a_{-12}^2 a_{20} b_{3,-1} - a_{-13} b_{02} b_{2,-1}^2 \\
&= a_{10} a_{-12}^2 b_{3,-1} b_{10} - a_{01} a_{-13} b_{2,-1}^2 b_{01} = a_{01}^2 a_{-13} b_{2,-1}^2 - a_{-12}^2 b_{3,-1} b_{10}^2 \\
&= a_{-12}^2 b_{20}^3 - a_{02}^3 b_{2,-1}^2 = a_{-12}^2 a_{20} b_{20}^2 - a_{02}^2 b_{02} b_{2,-1}^2 = a_{-12}^2 a_{20}^2 b_{20} - a_{02} b_{02}^2 b_{2,-1}^2 \\
&= a_{10}^2 a_{-12}^2 b_{3,-1} - a_{-13} b_{2,-1}^2 b_{01}^2 = a_{-12}^2 a_{20}^3 - b_{02}^3 b_{2,-1}^2 = a_{-12}^2 b_{3,-1}^2 b_{02} - a_{20} a_{-13}^2 b_{2,-1}^2 \\
&= a_{-12}^4 b_{3,-1}^3 - a_{-13}^3 b_{2,-1}^4 = a_{-12}^2 a_{02} b_{3,-1}^2 - a_{-13}^2 b_{20} b_{2,-1}^2 = a_{10} a_{01} - b_{10} b_{01} \\
&= a_{01} a_{-13}^2 b_{2,-1}^2 - a_{-12}^3 b_{3,-1}^2 b_{10} = a_{10} a_{-12}^3 b_{3,-1}^2 - a_{-13}^2 b_{2,-1}^3 b_{01}.
\end{aligned}$$

Proof: It is enough to show that the above equations (24) form a *Gröbner basis* of the ideal I_{sym} . To compute I_{sym} , one can use the *Symmetry Component Algorithm*, with any computer algebra system. Here we are using the specialized system *Macaulay* (13) which performs Gröbner basis calculations substantially faster than most general purpose symbolic calculation packages. Figure 1 shows the Macaulay session used to compute I_{sym} for system (23). To simplify notation, we renamed the variables a_{ij} and b_{ij} as follows: $x_1 = a_{10}, x_2 = a_{01}, x_3 = a_{-12}, x_4 = a_{20}, x_5 = a_{11}, x_6 = a_{02}, x_7 = a_{-1,3}, x_8 = b_{3,-1}, x_9 = b_{20}, x_{10} = b_{11}, x_{11} = b_{02}, x_{12} = b_{2,-1}, x_{13} = b_{10}, x_{14} = b_{01}$. \square

Recently some sufficient center conditions were obtained for the real systems with homogeneous nonlinearities of fourth and fifth degrees (6; 7). The following are the general complex forms of such systems.

$$\begin{aligned}
\dot{x} &= x(1 - a_{30}x^3 - a_{21}x^2y - a_{12}xy^2 - a_{03}y^3 - a_{-14}x^{-1}y^4), \\
\dot{y} &= -y(1 - b_{4,-1}x^4y^{-1} - b_{30}x^3 - b_{21}x^2y - b_{12}xy^2 - b_{0,3}y^3),
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
\dot{x} &= x(1 - a_{40}x^4 - a_{31}x^3y - a_{22}x^2y^2 - a_{13}xy^3 - a_{04}y^4 - a_{-15}x^{-1}y^5), \\
\dot{y} &= -y(1 - b_{5,-1}x^5y^{-1} - b_{40}x^4 - b_{31}x^3y - b_{22}x^2y^2 - b_{13}xy^3 - b_{04}y^4).
\end{aligned} \tag{26}$$

The following theorems give some center conditions for complex systems with homogeneous nonlinearities of fourth and fifth degrees, respectively.

THEOREM 10: *The symmetry component of the center variety of system (25) is defined by the following equations*

$$\begin{aligned}
0 &= a_{30}a_{03} - b_{30}b_{03} = a_{21}a_{12} - b_{21}b_{12} = a_{30}b_{12}^3 - a_{21}^3b_{03} \\
&= a_{30}a_{12}b_{12}^2 - a_{21}^2b_{21}b_{03} = a_{30}a_{12}^2b_{12} - a_{21}b_{21}^2b_{03} = a_{21}a_{03}b_{21}^2 - a_{12}^2 - b_{30}b_{12} \\
&= a_{21}^2a_{03}b_{21} - a_{12}b_{30}b_{12}^2 = a_{12}^3b_{30} - a_{03}b_{21}^3 = a_{21}^3a_{03} - b_{30}b_{12}^3 \\
&= a_{30}a_{12}^3 - b_{21}^3b_{03} = a_{30}a_{-1,4}b_{30}b_{12} - a_{21}a_{03}b_{4,-1}b_{03} \\
&= a_{30}^2a_{-1,4}b_{12} - a_{21}b_{4,-1}b_{03}^2 = a_{30}a_{-1,4}b_{21}^2 - a_{12}^2b_{4,-1}b_{03} \\
&= a_{21}a_{-1,4}b_{30}b_{21} - a_{12}a_{03}b_{4,-1}b_{12} = a_{30}a_{21}a_{-1,4}b_{21} - a_{12}b_{4,-1}b_{12}b_{03}
\end{aligned}$$

Figure 1: Macaulay computation for system(23)

$$\begin{aligned}
&= a_{12}a_{-1,4}b_{30}^2 - a_{03}^2b_{4,-1}b_{21} = a_{30}a_{12}a_{-1,4}b_{30} - a_{03}b_{4,-1}b_{21}b_{03} \\
&= a_{21}^2a_{-1,4}b_{30} - a_{03}b_{4,-1}b_{1,2}^2 = a_{21}a_{03}^2b_{4,-1} - a_{-1,4}b_{30}^2b_{12} \\
&= a_{30}^2a_{12}a_{-1,4} - b_{4,-1}b_{21}b_{03}^2 = a_{30}a_{21}^2a_{-1,5} - b_{4,-1}b_{12}^2b_{03} \\
&= a_{21}a_{-1,4}b_{21}^4 - a_{12}^4b_{4,-1}b_{12} = a_{21}^2a_{-1,4}b_{21}^3 - a_{12}^3b_{4,-1}b_{12}^2 \\
&= a_{31}^2a_{-1,4}b_{21}^3 - a_{12}^2b_{4,-1}b_{12}^3 = a_{21}^4a_{-1,4}b_{21} - a_{12}b_{4,-1}b_{12}^4 \\
&= a_{12}^5b_{4,-1} - a_{-1,4}b_{21}^5 = a_{21}^5a_{-1,4} - b_{4,-1}b_{12}^5 \\
&= a_{30}a_{-1,4}^2b_{30}^2b_{21} - a_{12}a_{03}^2b_{4,-1}^2b_{03} = a_{30}^2a_{-1,4}^2b_{30}b_{21} - a_{12}a_{03}b_{4,-1}^2b_{03}^2 \\
&= a_{30}^3a_{-1,4}^2b_{21} - a_{12}b_{4,-1}^2b_{03}^3 = a_{21}a_{-1,4}^3b_{30}^3 - a_{03}^3b_{4,-1}^2b_{12}
\end{aligned} \tag{27}$$

$$\begin{aligned}
&= a_{30}a_{21}a_{-1,4}^2b_{30}^2 - a_{03}^2b_{4,-1}^2b_{12}b_{03} = a_{30}^2a_{21}a_{-1,4}^2b_{30} - a_{03}b_{4,-1}^2b_{12}b_{03}^2 \\
&= a_{12}a_{03}^3b_{4,-1}^2 - a_{-1,4}^2b_{30}^3b_{21} = a_{30}^3a_{21}a_{-1,4}^2 - b_{4,-1}^2b_{12}b_{03}^3 \\
&= a_{30}a_{-1,4}^3b_{30}^4 - a_{03}^4b_{4,-1}^3b_{03} = a_{30}^2a_{-1,4}^3b_{30}^3 - a_{03}^3b_{4,-1}^3b_{03}^2 \\
&= a_{30}^3a_{-1,4}^3b_{30}^2 - a_{03}^2b_{4,-1}^3b_{03}^3 = a_{30}^4a_{-1,4}^3b_{30} - a_{03}b_{4,-1}^3b_{03}^4 \\
&= a_{03}^5b_{4,-1}^3 - a_{-1,4}^3 - b_{30}^5 = a_{30}^5a_{-1,4}^3 - b_{4,-1}^3b_{03}^5
\end{aligned}$$

Proof: Similar to the proof of Theorem 9. To simplify notation we rename the variables a_{ij} and b_{ij} : $x_1 = a_{30}, x_2 = a_{21}, x_3 = a_{12}, x_4 = a_{03}, x_5 = a_{-1,4}, x_6 = b_{4,-1}, x_7 = b_{30}, x_8 = b_{21}, x_9 = b_{12}, x_{10} = b_{03}$. Figure 2 is the *Macaulay* session used to find I_{sym} . \square

```

i2 : R = K[y_1..y_5,s,t,m,n, MonomialOrder => Lex];
i3 : I = ideal(s^m-1,t^n-1);
o3 : Ideal of R
i4 : Rbar = R/(I*R);
i5 : S = K[c,x_1..x_10, MonomialOrder => Lex];
i6 : f = map(Rbar, S, matrix(Rbar, {{n*m,y_1*s^3,y_2*s^2*t,y_3*s*t^2, y_4*t^3,y_5*m*t^4,
                                     y_5*s^4*n,y_4*s^3,y_3*s^2*t,y_2*s*t^2,y_1*t^3}}));
o6 : RingMap Rbar <--- S
i7 : J = ker f

o7 = ideal (x x  - x x  , x x  - x x  , x x  - x x  , x x x  - x x x  , x x x  - x x x  , x x x  - ...
            1 4      7 10   2 3      8 9      1 9      2 10   1 3 9      2 8 10   1 3 9      2 8 10   2 4 8
o7 : ideal of S

```

Figure 2: Macaulay computation for system(25)

THEOREM 11: *The symmetry component of the center variety of system (26) is defined by the following equations*

$$\begin{aligned}
0 &= a_{22} - b_{22} = a_{40}a_{04} - b_{40}b_{04} = a_{31}a_{13} - b_{31}b_{13} = a_{40}b_{13}^2 - a_{31}^2b_{04} \\
&= a_{40}a_{13}b_{13} - a_{31}b_{31}b_{04} = a_{31}a_{04}b_{31} - a_{13}b_{40}b_{13} = a_{13}^2b_{40} - a_{04}b_{31}^2 \\
&= a_{31}^2a_{04} - b_{40}b_{13}^2 = a_{40}a_{13}^2 - b_{31}^2b_{04} = a_{40}a_{-1,5}b_{31} - a_{13}b_{5,-1}b_{04} \\
&= a_{31}a_{-1,5}b_{40} - a_{04}b_{5,-1}b_{13} = a_{13}a_{04}b_{5,-1} - a_{-1,5}b_{40}b_{31} \\
&= a_{40}a_{31}a_{-1,5} - b_{5,-1}b_{13}b_{04} = a_{40}a_{-1,5}b_{40}a_{13} - a_{31}a_{04}b_{5,-1}b_{04} \\
&= a_{40}^2a_{-1,5}b_{13} - a_{31}b_{5,-1}b_{04}^2 = a_{31}a_{-1,5}b_{31}^2 - a_{13}^2b_{5,-1}b_{13} \\
&= a_{31}^2a_{-1,5}b_{31} - a_{13}b_{5,-1}b_{13}^2 = a_{13}a_{-1,5}b_{40}^2 - a_{04}^2b_{5,-1}b_{31} \\
&= a_{40}a_{13}a_{-1,5}b_{40} - a_{04}b_{5,-1}b_{31}b_{04} = a_{31}a_{04}2b_{5,-1} - a_{-1,5}b_{40}^2b_{13} \\
&= a_{13}^3b_{5,-1} - a_{-1,5}b_{31}^3 = a_{40}^2a_{13}a_{-1,5} - b_{5,-1}b_{31}b_{04}^2 = a_{31}^3a_{-1,5} - b_{5,-1}b_{13}^3 \\
&= a_{40}a_{-1,5}^2b_{40} - a_{04}^2b_{5,-1}^2b_{04} = a_{40}^2a_{-1,5}^2b_{40} - a_{04}b_{5,-1}^2b_{04}^2 \\
&= a_{04}^3b_{5,-1}^2 - a_{-1,5}^2b_{20}^3 = a_{40}^3a_{-1,5}^2 - b_{5,-1}^2b_{04}^3.
\end{aligned} \tag{28}$$

Proof: Similar to the proof of Theorem 9. \square

Thus we have presented an efficient algorithm to compute the symmetry component of the center variety. Up to now, the only known method for finding this component is due to Sibirsky. His algorithm is as follows.

1. He gives the formula (21) for an upper bound for the degrees of the irreducible invariants.
2. With this bound one can find all irreducible invariants by sorting.

In (36) a method is given to find all “elementary Lie invariants” (our Hilbert basis). There the problem is reduced to finding non-negative solutions of a Diophantine equation similar to our equation (17), which is done on a case-by-case basis by inspection.

Moreover neither Sibirsky nor Yi-Rong & Ji-Bin have an analog of our Theorem 1, which shows that the obtained invariants generate a prime ideal. But, as we will see in the next section, this fact is an important characterization of this subvariety of the center variety and is very helpful in the investigation of the cyclicity problem.

4. Applications to Cubic Systems

In recent years many studies have been devoted to investigating different subfamilies of the cubic system (23) (see, e.g., (4; 8; 9; 11; 24; 36; 38) and references therein). In the case when conditions (10) are satisfied system (23) is equivalent to system (22). It should be mentioned that the center-focus problem is much better investigated for the real cubic system (22) than for the general system (23).

In this section we will show that the results obtained above, together with additional tools from computational algebra, give a simple efficient way to compute the radical of the ideal of focus quantities and, therefore, to solve the cyclicity problem in those cases where this ideal is radical. As a rule it is easy to find rational parameterizations of components of center varieties. This fact, together with the following theorem and Theorem 2, gives an easy method of finding the radical of the ideal of focus quantities. For general rational parameterizations this is a difficult computational problem.

THEOREM 12: *If the variety $\mathbf{V}(J)$ of an ideal J of $\mathbb{C}[x_1, \dots, x_n]$ admits a rational parameterization*

$$x_i = \frac{f_i(t_1, \dots, t_m)}{g_i(t_1, \dots, t_m)}, \quad i = 1, \dots, n, \text{ and}$$

$$\mathbb{C}[x_1, \dots, x_n] \cap \langle 1 - tg, g_i(t_1, \dots, t_m)x_i - f_i(t_1, \dots, t_m) : i = 1, \dots, n \rangle = J$$

(where $g = g_1 g_2 \cdots g_n$), then the ideal J is a prime ideal of $\mathbb{C}[x_1, \dots, x_n]$.

Proof: It is sufficient to show that the ideal

$$H = \langle 1 - tg, g_i(t_1, \dots, t_m)x_i - f_i(t_1, \dots, t_m) : i = 1, \dots, n \rangle$$

is prime in $\mathbb{C}[x_1, \dots, x_n, t_1, \dots, t_m, t]$. Consider the ring homomorphism

$$\psi : \mathbb{C}[x_1, \dots, x_n, t_1, \dots, t_m, t] \longrightarrow \mathbb{C}(t_1, \dots, t_m),$$

defined by

$$x_i \longmapsto \frac{f_i}{g_i}, \quad t_l \longmapsto t_l, \quad t \longmapsto \frac{1}{g},$$

$i = 1, \dots, n, l = 1, \dots, m$. It is sufficient to prove that $\ker(\psi) = H$. It is clear that $H \subset \ker(\psi)$. We will show the other inclusion by induction.

Let us suppose that $h \in \ker(\psi)$ and h is linear in x , that is,

$$h = \sum_{i=1}^n \alpha_i(t_1, \dots, t_m, t)x_i + \alpha_0(t_1, \dots, t_m).$$

Then

$$\alpha_0 tg = - \sum_{i=1}^n \alpha_i f_i t \tilde{g}_i,$$

where $\tilde{g}_i = g/g_i$. Therefore

$$h = \sum_{i=1}^n \alpha_i (x_i g_i - f_i) t \tilde{g}_i + (1 - tg)h.$$

Hence, $h \in H$.

Assume now that for all polynomials of degree k in x_1, \dots, x_n and $h \in \ker(\psi)$, we have $h \in H$.

Let $h \in \ker(\psi)$ be of degree $k+1$ in x_1, \dots, x_n . We can write h in the form

$$h = \sum_{i=1}^n h_i(x_i, x_{i+1}, \dots, x_n, t_1, \dots, t_m, t) + h_0(t_1, \dots, t_m, t)$$

(here every term of h_i contains x_i). Consider the polynomial

$$u = \sum_{i=1}^n \frac{h_i}{x_i} (x_i g_i - f_i) t \tilde{g}_i + (1 - tg)h.$$

Then $u = h + v$, where

$$v = -t \sum_{i=1}^n f_i \tilde{g}_i \frac{h_i}{x_i} - tgh_0.$$

Since $h, u \in \ker(\psi)$ we get that $v \in \ker(\psi)$. Then, by the induction hypothesis, $v \in H$ and, hence, $h \in H$. \square

We now apply the results obtained so far to the investigation of the cyclicity problems in some specific cases. First consider the systems with homogeneous quadratic and cubic nonlinearities,

$$\begin{aligned}\dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \\ \dot{y} &= -(y - b_{10}xy - b_{01}y^2 - b_{2,-1}x^2)\end{aligned}\tag{29}$$

and

$$\begin{aligned}\dot{x} &= x - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{-13}y^3, \\ \dot{y} &= -(y - b_{02}y^3 - b_{11}xy^2 - b_{20}x^2y - b_{3,-1}x^3),\end{aligned}\tag{30}$$

correspondingly.

In the case when the conditions (10) hold and the linear perturbations also are taking into account the system (29) corresponds to the real system on the plane (u, v) , $x = u + iv$

$$i\dot{x} = i\lambda x - x - a_{10}x^2 - a_{01}x\bar{x} - a_{-12}\bar{x}^2\tag{31}$$

and the system (30) corresponds to

$$i\dot{x} = i\lambda x - x - a_{20}x^3 - a_{11}x^2\bar{x} - a_{02}x\bar{x}^2 - a_{-13}\bar{x}^3,\tag{32}$$

where $\lambda \in \mathbb{R}$.

For the first time the cyclicity of the origin of the system (31) was investigated by Bautin (3) and of the system (32) by Sibirsky (30) (later on another proofs were obtained by Żołądek (37; 38) and Yakovenko (35)). They proved that the following statement holds.

THEOREM 13: *The cyclicity of the origin of system (31) equals 3 (3; 37; 35) and the cyclicity of the origin of system (32) equals 5 (30; 38).*

The crucial and the most difficult part of the proofs of this theorem is the following statement (see e.g. (3; 22; 35; 37) for detail derivation Theorem 13 from Proposition 3).

PROPOSITION 3: *1) The first three focus quantities generate the ideal of focus quantities of systems (29) and (31) with $\lambda = 0$.*

2) The first five focus quantities generate the ideal of focus quantities of systems (30) and (32) with $\lambda = 0$.

Bautin proved Proposition 3 for the real quadratic system (31) in the Kapteyn form. The proof is quite complicated, because the ideal of focus quantities is not radical in this case. A simpler proof on Bautin's way was given by Yakovenko (35).

Żołądek (37; 38) found a new way. He proved the Proposition 3 for the ring of polynomials, which are invariant under the action of the rotation group and showed that the ideal of focus quantities is radical in this ring.

We will show that with Theorems 1 and 12 the proof of the celebrated Bautin

theorem, as well as the treatment of the cyclicity problem for the system (30), becomes straightforward, using only basic knowledge of computer algebra.

Consider first the system (30). Let $I^{(c)} = \langle g_{11}, g_{22}, \dots \rangle$ be the ideal generated by all focus quantities of this system (the so-called *Bautin ideal*) and let $I_k^{(c)} = \langle g_{11}, g_{22}, \dots, g_{kk} \rangle$ be the ideal generated by the first k focus quantities.

Computing the first five focus quantities by means of the algorithm given in (21), and then reducing them, we find that

$$\begin{aligned} g_{11} &= a_{11} - b_{11}; \\ g_{22} &= a_{20}a_{02} - b_{02}b_{20}; \\ g_{33} &= (3a_{20}^2a_{-13} + 8a_{20}a_{-13}b_{20} + 3a_{02}^2b_{3,-1} \\ &\quad - 8a_{02}b_{02}b_{3,-1} - 3a_{-13}b_{20}^2 - 3b_{02}^2b_{3,-1})/8; \\ g_{44} &= (-9a_{20}^2a_{-13}b_{11} + a_{11}a_{-13}b_{20}^2 + 9a_{11}b_{02}^2b_{3,-1} - a_{02}^2b_{11}b_{3,-1})/16; \\ g_{55} &= (-9a_{20}^2a_{-13}b_{02}b_{20} + a_{20}a_{02}a_{-13}b_{20}^2 + 9a_{20}a_{02}b_{02}^2b_{3,-1} + \\ &\quad 18a_{20}a_{-13}^2b_{20}b_{3,-1} + 6a_{02}^2a_{-13}b_{3,-1}^2 - a_{02}^2 + \\ &\quad b_{02}b_{20}b_{3,-1} - 18a_{02}a_{-13}b_{02}b_{3,-1}^2 - 6a_{-13}^2b_{20}^2b_{3,-1})/36. \end{aligned}$$

Furthermore, the variety of the ideal I coincides with the variety of the first five focus quantities. More precisely, the following result holds.

THEOREM 14: *The center variety $\mathbf{V}(I^{(c)})$ of the system (30) consists of the three irreducible components:*

$$\mathbf{V}(I^{(c)}) = \mathbf{V}(I_5^{(c)}) = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) \cup \mathbf{V}(J_3), \quad (33)$$

where $J_1 = \langle a_{11} - b_{11}, 3a_{20} - b_{20}, 3b_{02} - a_{02} \rangle$, $J_2 = \langle a_{11}, b_{11}, a_{20} + 3b_{20}, b_{02} + 3a_{02}, a_{-13}b_{3,-1} - 4a_{02}b_{20} \rangle$ and $J_3 = \langle a_{20}^2a_{-13} - b_{3,-1}b_{02}^2, a_{20}a_{02} - b_{20}b_{02}, a_{20}a_{-13}b_{20} - a_{02}b_{3,-1}b_{02}, a_{11} - b_{11}, a_{02}^2b_{3,-1} - a_{-13}b_{20}^2 \rangle$.

Proof: It is easy to check (using, for example, the radical membership test, see e.g. (5)) that $\mathbf{V}(I_5^{(c)}) = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) \cup \mathbf{V}(J_3)$. Thus we only have to show that $g_{mm}|_{\mathbf{V}(J_k)} \equiv 0$ for all $m > 0$ and $k = 1, 2, 3$.

Indeed, the systems corresponding to the points of $\mathbf{V}(J_1)$ are Hamiltonian.

For the variety $\mathbf{V}(J_2)$ in the case $a_{-13} = (4a_{02}b_{20})/b_{3,-1}$ one can easily find an invariant conic l_1 and invariant cubic l_2 and check that the system has the first integral

$$\Phi = l_1^3 l_2^{-2}, \quad (34)$$

defined on $\mathbf{V}(J_2) \setminus \mathbf{V}(H)$, where $H = \langle b_{20}b_{3,-1}^2, b_{3,-1}^2(a_{02}b_{3,-1}^2 + 4b_{20}^3) \rangle$. Computing the quotient of the ideals (e.g. by means of the algorithm from (5)) we get $J_2 : H = J_2$. Therefore,

$$\overline{\mathbf{V}(J_2) \setminus \mathbf{V}(H)} = \mathbf{V}(J_2).$$

This implies that the system (30) has a center on the whole component $\mathbf{V}(J_2)$.

Finally, according to Theorem 9, $\mathbf{V}(J_3)$ is the symmetry component of the center variety.

The irreducibility of the components $\mathbf{V}(J_1)$ and $\mathbf{V}(J_2)$ is obvious, and $\mathbf{V}(J_3)$ is irreducible because due to Theorems 2 and 9 the ideal J_3 is prime. \square

A very similar theorem is proven also in (36) (and in (12; 38) for the real case). However, our theorem also shows that (33) is the irreducible decomposition of the center variety.

It is worthwhile to mention that, if one presents the solution of the center problem for a system of the type (9) writing out the irreducible decomposition of a center variety, then the answer is unique. Otherwise it can happen that the center conditions for the same system obtained by different authors can look very different. In fact the center conditions for system (30) were first obtained by Sadosky (27). He found eight such sets but they look very different from the ones presented in Theorem 14. However computing the Zariski closure of the sets given by Sadosky and taking intersections of the corresponding ideals using standard algorithms from computational algebra (see below) we get the ideal $I_5^{(c)}$. This implies that his center conditions coincide with the three components given in Theorem 14.

Recall that we can use the following algorithm to compute the intersection of two ideals (5). Let $I = \langle f_1, \dots, f_r \rangle$ and $J = \langle h_1, \dots, h_s \rangle$ be ideals in $k[x_1, \dots, x_n]$. Compute a Gröbner basis for the ideal

$$\langle tf_1, \dots, tf_r, (1-t)h_1, \dots, (1-t)h_s \rangle \subset k[t, x_1, \dots, x_n]$$

using a lexicographic term order with t greater than the x_i . Those elements of this basis which do not contain the variable t will form a basis of $I \cap J$.

For the case of quadratic system one can easily compute the three first focus quantities and get

$$\begin{aligned} g_{11}^{(q)} &= a_{10}a_{01} - b_{10}b_{01}, \\ g_{22}^{(q)} &= a_{10}a_{-12}b_{10}^2 - a_{01}^2b_{01}b_{2,-1} - \frac{2}{3}(a_{-12}b_{10}^3 - a_{01}^3b_{2,-1}) - \frac{2}{3}(a_{01}b_{01}^2b_{2,-1} - a_{10}^2a_{-12}b_{10}), \\ g_{33}^{(q)} &= \frac{5}{12}(a_{01}^2a_{10}b_{01}^2b_{2,-1} - a_{10}^2a_{-12}b_{01}b_{10}^2) - \frac{5}{48}(a_{01}^3b_{01}b_{10}b_{2,-1} - a_{01}a_{10}a_{-12}b_{10}^3) \\ &\quad + \frac{5}{8}(a_{01}a_{-12}b_{01}^2b_{2,-1}^2 - a_{10}^2a_{-12}^2b_{10}b_{2,-1}) - \frac{5}{16}(a_{01}^2a_{-12}b_{01}b_{2,-1}^2 - a_{10}a_{-12}^2b_{10}^2b_{2,-1}). \end{aligned}$$

Here and below the upper index (q) means that we are speaking about the quadratic system.

Using the equations above and Theorems 1 and 12 it is easy to see (similarly to the case of the system (30)) that the following theorem holds (see (10; 12; 31; 37) for details).

THEOREM 15: *The center variety of the system (29) consists of four irreducible components:*

1. $\mathbf{V}(J_1^{(q)})$, where $J_1^{(q)} = \langle 2a_{10} - b_{10}, 2b_{01} - a_{01} \rangle$,
2. $\mathbf{V}(J_2^{(q)})$, where $J_2^{(q)} = \langle a_{01}, b_{10} \rangle$,

3. $\mathbf{V}(J_3^{(q)})$, where $J_3^{(q)} = \langle 2a_{01} + b_{01}, a_{10} + 2b_{10}, a_{01}b_{10} - a_{-12}b_{2,-1} \rangle$,
4. $\mathbf{V}(J_4^{(q)}) = \langle f_1, f_2, f_3, f_4, f_5 \rangle$, where $f_1 = a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3$, $f_2 = a_{10} a_{01} - b_{01} b_{10}$, $f_3 = a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3$, $f_4 = a_{10} a_{-12} b_{10}^2 - a_{01}^2 b_{2,-1} b_{01}$, $f_5 = a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2$.

To our knowledge the notion of the center variety was introduced in the literature on the center problem very recently by Żołądek in (39), where he also gives Theorem 15. But he writes out the fourth component in the form

$$f_1 = f_2 = 0.$$

It is not precise because, using Theorem 12, one can see that

$$\mathbf{V}(\langle f_1, f_2 \rangle) = \mathbf{V}(J_4^{(q)}) \cup \mathbf{V}(J_2^{(q)}).$$

LEMMA 2: *The ideals $I^{(q)}$ and $I^{(c)}$ are radical ideals in $\mathbb{C}[a_{10}, a_{01}, a_{-12}, b_{2,-1}, b_{10}, b_{01}]$ and $\mathbb{C}[a_{20}, a_{11}, \dots, b_{02}]$, respectively.*

Proof: Computing the intersection of the ideals J_k we find

$$I_5^{(c)} = J_1 \cap J_2 \cap J_3.$$

Hence $I_5^{(c)}$ is radical because due to Theorem 12 J_1, J_2 are prime, and according to Theorems 2 and 9 J_3 is prime as well.

Similarly, for quadratic system we easily check that, for $1 \leq i \leq 4$, the ideals $J_i^{(q)}$ are prime and

$$I_3^{(q)} = \cap_{i=1}^4 J_i^{(q)}. \quad (35)$$

This yields that the ideal of focus quantities of quadratic system is a radical ideal. \square

Proof: (Proposition 3) $\mathbf{V}(I^{(q)}) = \mathbf{V}(I_3^{(q)})$ ($\mathbf{V}(I^{(c)}) = \mathbf{V}(I_5^{(c)})$) and the ideal $I_3^{(q)}$ ($I_5^{(c)}$) is a radical ideal. Therefore $I^{(q)} = I_3^{(q)}$ ($I^{(c)} = I_5^{(c)}$). \square

In (16), we study the following system.

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-13}y^3, \\ \dot{y} &= -(y - b_{01}y^2 - b_{10}xy - b_{3,-1}x^3). \end{aligned} \quad (36)$$

Using the methods of this paper, we compute the ideal of the center variety of (36) which turns not to be radical. Moreover, we prove the following theorem.

THEOREM 4.1: *The cyclicity of the origin of the system*

$$i\dot{x} = x - a_{10}x^2 - a_{01}x\bar{x} - a_{-13}\bar{x}^3,$$

is less than or equal to 5.

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